Joe Dinius
HW2

MATH 527A
November 28, 2008

1. A function $f : (M, d) \to (M', d')$ is *continuous*, if $x_n \to x$ in (M, d) implies that $f(x_n) \to f(x)$ in (M',d').

(a) Propose an $\epsilon - \delta$ definition for continuity and prove that your definition agrees with the above characterization.

Solution. Continuity in metric spaces requires definition before beginning the problem. A sequence x converges if $\exists x \in M$ s.t. $\forall \epsilon > 0$ $\exists N$ s.t. $\forall n > N$, $d(x_n, x) < \epsilon$.

Let (M, d) and (M', d') be metric spaces. A function $f : (M, d) \to (M', d')$ is continuous at a point $x_0 \in M$ if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $d(x, x_0) < \delta \implies d'(f(x), f(x_0)) < \epsilon$. If f is continuous at all points $x_i \in X$, f is said to be *continuous on* X.

I now need to prove that the definition is consistent.

Proof. Suppose $f : (M, d) \to (M', d')$ is continuous. Let x_n be a sequence in M s.t. $x_n \to x$. Since f is continuous, we know that given any $\epsilon > 0$, we can find a $\delta > 0$ such that $d(x_n, x) < \delta \implies d'(f(x_n), f(x)) < \epsilon$. But, using the definition for convergence in a metric space, $x_n \to x$, so $\exists N$ s.t. $\forall n > N, d(x_n, x) < \delta$. Therefore, $d'(f(x_n), f(x)) < \epsilon \implies f(x_n) \to f(x)$.

From the definition of convergence of $x_n \to x$ in (M, d), for this and every choice of $\delta > 0$, $\exists N$ s.t. for $n > N$,

$$d(x_n, x) < \delta.$$

Then the following holds for every $\epsilon > 0$ and there exists N s.t. for $n > N$

$$d'(f(x_n), f(x)) < \epsilon.$$

This is the definition of convergence for $f(x_n) \to f(x)$ in (M', D'). Now since $x_n \to x$ in (M, d) implies that $f(x_n) \to f(x)$ in (M', d'), the function f is continuous. So, the $\epsilon - \delta$ definition does imply continuity.

Therefore the definition is consistent. □

(b) d and d' are two metrics on M. The metric d' is said to be *coarser* than d (equivalently, d is *finer* than d') if every sequence that converges in (M, d) also converges in (M, d'). Show that a metric d' on M is coarser than metric d iff the identity map from (M, d) to (M, d') is continuous. What is the $\epsilon - \delta$ characterization equivalent to the above definition for d' being coarser than d?

Solution. The $\epsilon - \delta$ definition is the same as given in class: d' is coarser than d if $\forall \epsilon > 0$ $\exists \delta > 0$ s.t. $d(x, y) < \delta \implies d'(x, y) < \epsilon$.

$(\implies)$ Assume d' is coarser than d and $a_n \to a$ in (M,d) and in (M,d'). We can say that $a_n \to a$ in $(M,d) \implies id(a_n) = a_n \to a = id(a)$ in (M,d'); which is the definition of continuity on (M,d').

$(\Longleftarrow)$ Let $x_n \in M$ and id is continuous. The function id continuous implies $x_n \to x \in (M,d) \implies x_n \to x \in (M,d')$. This statement directly implies d' coarser than d.

Therefore the claim is true.

(c) x_0 is a given point in a metric space (M,d). Show that the function $f : M \to \mathbb{R}$ defined by $f(x) = d(x,x_0)$ is a continuous function.

Solution. Consider first the notion of what it is to be a metric space, i.e. the triangle inequality.

$$\begin{aligned} d(a,x_0) &\leq d(a,b) + d(b,x_0) \\ d(a,x_0) &- d(b,x_0) \leq d(a,b) \\ d(b,x_0) &\leq d(b,a) + d(a,x_0) \\ d(b,x_0) &- d(a,x_0) \leq d(b,a) = d(a,b) \\ \implies |d(a,x_0) &- d(b,x_0)| \leq d(a,b) \end{aligned}$$

for arbitrary $a,b \in M$.

Carrying the definition from (a), we get

$$|f(a) - f(b)| = |d(a,x_0) - d(b,x_0)| \leq d(a,b) < \epsilon.$$

Since b is arbitrary, f is continuous on M.

(d) Show that the mapping on sequences induced by the function $x_n \mapsto e^{x_n} - 1$ is a continuous map from $l^p(\mathbb{R},\mathbb{N})$ into itself.

Solution. First we check that if $x \in l^p$ then $f(x) \in l^p$. We have the sum

$$||f(x)||_p^p = \sum_{i=1}^{\infty} |e^{x_i} - 1|^p.$$

Analyze each term of the sum, and via the mean value theorem ($f(a) - f(b) = f'(c)(a-b)$ for $c \in (a,b)$),

$$|e^{x_i} - e^0| = |(x_i - 0)e^c| \leq |x_i| \sup_{1 \leq j \leq \infty} (e^{|x_j|} + 1) = M|x_i|$$

then we can bound $||f(x)||_p^p$ using that M is a constant and is bounded since each x_j is bounded by $||x||_\infty \leq ||x||_p$,

$$||f(x)||_p^p \leq \sum_{i=1}^{\infty} |e^{x_i} - 1| \leq M^p \sum_{i=1}^{\infty} |x_i|^p = M^p ||x_i||_p^p < \infty.$$

After taking the p-root we obtain that $f(x) \in l^p$ since $||f(x)||_p \leq M||x||_p < \infty$. Note that, for two real numbers a and b, we have

$$|e^a - e^b| = |a-b|e^c \leq e^{(|a|+|b|)}|a-b|$$

where the existence of an appropriate c between a and b is guaranteed by the mean value theorem.

Consequently, for $x, y \in l^p$, we have

$$\begin{aligned}\sum_{i=1}^{n} |e^{x_i} - 1 - e^{y_i} + 1|^p &\leq \sum_{i=1}^{n} e^{p(|x_i|+|y_i|)}|x_i - y_i|^p \\ &\leq \sum_{i=1}^{n} e^{p(||x||_\infty + ||y||_\infty)}|x_i - y_i|^p \\ &\leq e^{p(||x_p||+||y||_p)}||x-y||_p^p.\end{aligned}$$

Give $x \in l^p$ and $\epsilon > 0$, if we set $\delta = \min(1, e^{-2||x||_p - 1}\epsilon)$, it is easy to see that $||x - y||_p < \delta$ implies that $||y||_p \leq ||x||_p + 1$, and also that $e^{p(||x_p||+||y||_p)}||x-y||_p < \epsilon$, which shows that the mapping is continuous.

(e) Is the mapping on sequences induced by the function $x_n \mapsto \sqrt{x_n}$ a continuous map from $l^1(\mathbb{R}, \mathbb{N})$ into $l^2(\mathbb{R}, \mathbb{N})$? Prove or give a counter example.

The conjecture is that the map is continuous.

Proof. Since $\sum(\sqrt{|x_n|})^2$ converges iff $\sum |x_n|$ converges, it follows that the mapping is from l^1 into l^2.

Note that

$$\begin{aligned}(\sqrt{a} - \sqrt{b})^2 &= (\sqrt{a} - \sqrt{b})^2 \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} \\ &= |a-b|\left|\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right|.\end{aligned}$$

Applying this result to two l^1 sequences x and y, we see that

$$\sum_{i=1}^{N}(\sqrt{|x_i|} - \sqrt{|y_i|})^2 \leq \sum_{i=1}^{N} ||x_i| - |y_i|| \leq ||x-y||_1.$$

This is true uniformly on N, so

$$||\sqrt{|x_i|} - sqrt|y_i|||_2 \leq \sqrt{||x-y||_1}$$

from which it is easy to infer that the mapping is continuous.

Therefore, the mapping $x_n \to \sqrt{x_n}$ is continuous into $l^2(\mathbb{R}, \mathbb{N})$. □

2. **Young's inequality**.

(a) Ex. 1.3.39 from Flaschka:
Here is a generalization of Young's inequality (for some people, it is the Young inequality). Let f be continuous and strictly increasing on $[0,\infty)$, with $f(0)=0$. Let f^{-1} be the inverse function. Then for $a, b > 0$,

$$ab \leq \int_0^a f(t)dt + \int_0^b f^{-1}(t)dt, \tag{1}$$

and equality hold iff $b = f(a)$.

For $f(t) = t^{p-1}$, (1) becomes Young's inequality. Adapt a pictorial proof to explain why (1) should be true.

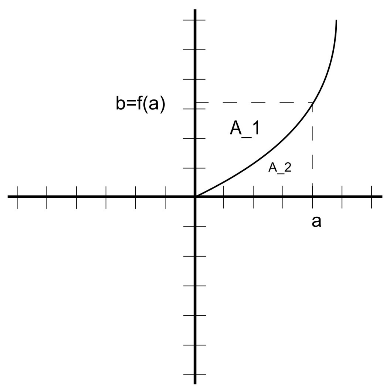

Figure 1: Picture for Young's inequality generalization

Proof. For any non-negative function $f : [0,a] \to \mathbb{R}$, $\int_0^a f(t)dt$ is the area underneath the curve, A_2, as shown in Figure 1. Similarly, for $f^{-1} : [0,b] \to \mathbb{R}$, $\int_0^b f^{-1}(t)dt$ is the area A_1. It follows that $A_1 + A_2 \geq ab$:

$$\implies ab \leq \int_0^a f(t)dt + \int_0^b f^{-1}(t)dt.$$

It is clear from the picture that if $b = f(a)$, we have equality, that is

$$ab = \int_0^a f(t)dt + \int_0^b f^{-1}(t)dt.$$

Similarly, if equality given, $b = f(a)$. This completes the "proof". $\square$

(b) Show that, $\forall a, b \geq 0$, we have

$$\exp(a) + (1+b)\log(1+b) \geq (1+a)(1+b).$$

When do we have equality in this result?

Solution. Use part (a), with $f(t) = e^t$, except at $t = 0$ where $f(0) = 0$ and $f^{-1}(t) = \log t$, since f is strictly increasing on all of $\mathbb{R}$. Therefore, consider

$$mn \leq \int_0^m f(t)dt + \int_0^n f^{-1}(t)dt$$

with $m = 1 + a$ and $n = 1 + b$.

Henceforth,

$$\begin{aligned}
(1+a)(1+b) &\leq \int_0^{1+a} f(t)dt + \int_0^{1+b} f^{-1}(t)dt \\
&= \int_0^{1+a} e^t dt + \int_0^{1+b} \log t dt \\
&= f(t)\Big|_0^{1+a} + (t\log t - t)\Big|_0^{1+b} \\
&= e^{1+a} - 0 + (1+b)\log(1+b) - (1+b) \\
&\leq e^a + (1+b)\log(1+b)
\end{aligned}$$

since $a, b > 0$. When $a = 0, b = f(a) = 0$ we have equality.

3. $x \in \mathbb{R}^{\mathbb{N}}$ is a sequence. q satisfies $1 \leq q \leq \infty$. For every $y \in l^q(\mathbb{R}, \mathbb{N})$, we know that the sequence $x_i y_i$. is absolutely summable, and

$$\left|\sum_{i=1}^{\infty} x_i y_i\right| \leq C||y||_q$$

for a constant C. Show that $x \in l^p(\mathbb{R}, \mathbb{N})$ where $p = q/(q-1)$.

Solution. Let $y_n = |x_n^N|^{p-1}$ where $x_n^N = \{x_n : \exists N \text{ s.t. } x_m = 0 \ \forall m > N\}$. Clearly, $y \in l^q(\mathbb{R}, \mathbb{N})$ since all terms past N are 0. Therefore,

$$\left|\sum_{i=1}^{N} x_i y_i\right| = \sum_{i=1}^{N} |x_i|^p = ||x^N||_p^p.$$

Thus,

$$\begin{aligned}
||x^N||_p^p \leq C||y||_p &= C||x^N||_p^{p-1} \\
\implies ||x^N||_p &\leq C < \infty.
\end{aligned}$$

Since the above argument is independent of N, we can state the following

$$||x||_p \leq C < \infty.$$

Therefore $x \in l^p(\mathbb{R}, \mathbb{N})$.

4. Exercises 1.3.31, 1.3.32 and 1.3.33 of Flaschka's notes.

(1.3.31) Inequality

$$||\mathbf{y}||_2 \leq C||\mathbf{x}||_2 \tag{2}$$

can be used to prove that matrix multiplication is continuous: if $\mathbf{x}^{(n)} \to \mathbf{x}$ is the usual Euclidean sense, then $\mathbf{A}\mathbf{x}^{(n)} \to \mathbf{A}\mathbf{x}$. Give the details.

Proof. Let $\mathbf{x} \in \mathbb{R}^n$ and let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Suppose $\mathbf{x}^{(n)}$ converges to $\mathbf{x}$ in the standard Euclidean sense; that is, $||\mathbf{x}^{(n)} - \mathbf{x}||_2 \to 0$. We want to show that $||\mathbf{y}^{(n)} - \mathbf{y}||_2 \to 0$, where $\mathbf{y} = \mathbf{A}\mathbf{x}$ and $\mathbf{y}^{(n)} = \mathbf{A}\mathbf{x}^{(n)}$. This follows very straightforwardly,

$$||\mathbf{A}\mathbf{x}^{(n)} - \mathbf{A}\mathbf{x}||_2 = ||\mathbf{A}(\mathbf{x}^{(n)} - \mathbf{x})||_2 \leq ||\mathbf{A}||_2||\mathbf{x}^{(n)} - \mathbf{x}||_2 = C||\mathbf{x}^{(n)} - \mathbf{x}||_2 \to 0$$

since $\mathbf{A}$ is of finite rank, the 2-norm is finite, call it C

$$= \sqrt{\sum_{i=0}^{n}\sum_{j=0}^{n} |a_{ij}|^2}$$

Therefore, matrix multiplication is continuous. □

(1.3.32) Inequality (2) remains true even if $\mathbf{x}$ belongs to $l^2(\mathbb{R}, \mathbb{N})$, as long as the *infinite* matrix $\mathbf{A}$ satisfies

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |a_{ij}|^2 < \infty.$$

If this is true, then (2) shows that $\mathbf{y}$ also belongs to $l^2(\mathbb{R}, \mathbb{N})$, and that multiplication by this infinite matrix is again continuous. Write out the details.

Solution. This follows almost verbatim from above. Assume $\mathbf{x} \in l^2(\mathbb{R}, \mathbb{N})$. Therefore $||\mathbf{x}||_2 < \infty$. By Hölder's equality,

$$||\mathbf{A}\mathbf{x}||_2 \leq ||\mathbf{A}||_2||\mathbf{x}||_2$$

We are given that

$$\begin{aligned} ||\mathbf{A}||_2^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |a_{ij}|^2 < \infty \\ \implies ||\mathbf{A}||_2 < \infty \end{aligned}$$

and thus

$$||\mathbf{A}\mathbf{x}||_2 < \infty \implies \mathbf{y} = \mathbf{A}\mathbf{x} \in l^2(\mathbb{R}, \mathbb{N}).$$

Continuity follows immediately from (1.3.31).

(1.3.33) Let $\mathbf{A} = (a_{ij})$ be a real $n \times n$ matrix. Let $\mathbf{x} \in \mathbb{R}^n$, and let $\mathbf{y} = \mathbf{A}\mathbf{x}$. Set

$$\begin{aligned} M &= \sup_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \text{ and} \\ N &= \sup_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|. \end{aligned}$$

(M and N are called the *maximum row sum of* $\mathbf{A}$ and the *maximum column sum of* $\mathbf{A}$, respectively.) Let $1/p + 1/q = 1, 1 < p, q < \infty$. Show that

$$||\mathbf{y}||_q \le M^{1/p} N^{1/q} ||\mathbf{x}||_q.$$

Here $||\mathbf{x}||_q$ is the "l^q-length of $\mathbf{x}$",

$$||\mathbf{x}||_q \equiv \sqrt[q]{\sum_{j=1}^{n} |x_j|^q}.$$

(Hint: write

$$|y_i| \le \sum_{j=1}^{n} |a_{ij}|^{1/p} |a_{ij}|^{1/q} |x_j|,$$

and use Hölder's inequality.)

Solution. Begin by looking at the i^{th} entry of $\mathbf{y}$, as the hint indicates

$$|y_i| \le \sum_{j=1}^{n} |a_{ij}^{1/p} a_{ij}^{1/q} x_j|.$$

Follow the steps to get the maximal row sum piece:

$$\begin{aligned} |y_i| &\le \sum_{j=1}^{n} |a_{ij}^{1/p} (a_{ij}^{1/q} x_j)| \\ &\le ||a_{ij}||_1^{1/p} ||a_{ij}^{1/q} x_j||_q \text{ by Hölder's inequality} \\ &\le M^{1/p} ||a_{ij}^{1/q} x_j||_q. \end{aligned}$$

I can use the same argument as above, summing over over i to get

$$\begin{aligned} ||\mathbf{y}||_q &\le M^{1/p} \sum_{i=1}^{n} |a_{ij}^{1/q} x_i| \\ &\le M^{1/p} ||a_{ij}||_1^{1/q} ||\mathbf{x}||_q \\ &\le M^{1/p} N^{1/q} ||\mathbf{x}||_q, \end{aligned}$$

which is the desired result.

5. Exercises 1.4.39 and 1.4.40 from Flaschka's notes:

(1.4.39) (A. Torchinsky) Let $f \in L_0^2([0,\pi])$. Is it possible to have simultaneously

$$\int_0^\pi (f(t) - \sin t)^2 dt \leq 4/9 \quad \text{and} \quad \int_0^\pi (f(t) - \cos t)^2 dt \leq 1/9?$$

Solution. Suppose such an f exists. Therefore $f(t) - \sin t$ and $f(t) - \cos t$ are both in $L^2([0,\pi])$ with norms $\leq \frac{2}{3}$ and $\frac{1}{3}$, respectively. Since $L^2([0,\pi])$ is a metric space, we can use the triangle inequality and symmetry to make the following comparison

$$\begin{aligned} ||(f(t) - \sin t) + (\cos t - f(t))||_2 &\leq ||f(t) - \sin t||_2 + ||\cos t - f(t)||_2 \\ &= ||f(t) - \sin t||_2 + ||f(t) - \cos t||_2 \\ &\leq \frac{2}{3} + \frac{1}{3} = 1. \end{aligned}$$

This would imply

$$||\cos t - \sin t||_2 = \sqrt{\int_0^\pi (\cos t - \sin t)^2 dt} \leq 1.$$

However, carrying out the integration, $||\cos t - \sin t||_2 = \sqrt{\pi} \approx 1.7725 \not\leq 1$, which is a contradiction.

Therefore, such a function f cannot exist that satisfies the given inequalities.

(1.4.40) Since $e^{-t}/t \leq e^{-t}$ when $t \geq 1$, we can estimate

$$\int_1^\infty \frac{e^{-t}}{t} dt \leq \int_1^\infty e^{-t} dt \sim .368\ldots .$$

Show how to get the better estimate

$$\int_1^\infty \frac{e^{-t}}{t} dt \leq \frac{e^{-1}}{\sqrt{2}} \sim .26\ldots .$$

Solution. Use Hölder's inequality, $p = q = 2$, to get

$$\begin{aligned} \int_1^\infty \frac{e^{-t}}{t} dt &\leq ||e^{-2t}||_2 ||t^{-2}||_2 \\ &= \left(\int_1^\infty e^{-2t} dt\right)\left(\int_1^\infty t^{-2} dt\right) \\ &= \left(\sqrt{\frac{e^{-2}}{2}}\right)\sqrt{1} = \frac{e^{-1}}{\sqrt{2}}, \end{aligned}$$

which is the desired result.

6. Assume that a, b and c are such that $1 \leq a < c \leq \infty$. Also, $f \in C(0, \infty)$.

(a) If f is a dimensional quantity with dimensions Q and x has dimensions L, what are the dimensions of the L^p norm of f?

Solution. I can write the L^p norm as

$$\left(\int |f(x)|^p dx \right)^{1/p}$$

which has dimensions

$$\left[(Q^p \cdot L)^{1/p} \right] = \left[Q \cdot L^{1/p} \right].$$

(b) Using dimensional analysis, determine exponents α and β such that

$$||f||_b \leq C||f||_a^\alpha ||f||_c^\beta.$$

Solution. Use the result from part (a) and apply to the inequality given. Therefore,

$$\begin{aligned} \left[Q \cdot L^{1/b} \right] &\leq C\left[Q \cdot L^{1/a} \right]^\alpha \left[Q \cdot L^{1/c} \right]^\beta \\ &= C\left[Q^{\alpha+\beta} \cdot L^{\frac{\alpha}{a}+\frac{\beta}{b}} \right]^\alpha \end{aligned}$$

This leads to the coupled algebraic equations

$$\begin{aligned} \alpha + \beta &= 1 \\ \frac{\alpha}{a} + \frac{\beta}{b} &= \frac{1}{b}. \end{aligned}$$

whose solution is

$$\begin{aligned} \alpha &= \frac{c(a-b)}{b(a-c)} \\ \beta &= \frac{a(b-c)}{b(a-c)}. \end{aligned}$$

(c) Prove that the above inequality is indeed true, with $C = 1$.

Proof. From above, we have $b = b\alpha + b\beta$ which leads to

$$\int |f(x)|^b dx = \int |f(x)^{b\alpha} f(x)^{b\beta}| dx.$$

I would like to get the above relationship into some form that I can apply Hölder's inequality to. Notice that, given the second equation from part (b) of this exercise to show that $1 = \frac{b\alpha}{a} + \frac{b\beta}{c}$, so the two terms are conjugates. Apply Hölder's inequality using $p = \frac{b\alpha}{a}$ and $q = \frac{b\beta}{c}$ to get

$$\begin{aligned}
\int |f(x)|^b dx &= \int |f(x)^{b\alpha} f(x)^{b\beta}| dx \\
&\leq \left(\int |f(x)^{b\alpha}|^{\frac{b\alpha}{a}} dx \right)^{\frac{a}{b\alpha}} \left(\int |f(x)^{b\beta}|^{\frac{b\beta}{c}} dx \right)^{\frac{c}{b\beta}} \\
&= \left(\int |f(x)^{\frac{b^2\alpha^2}{a}}| dx \right)^{\frac{a}{b\alpha}} \left(\int |f(x)^{\frac{b^2\beta^2}{c}}| dx \right)^{\frac{c}{b\beta}} \\
&\leq \left(\int |f(x)|^a dx \right)^{\frac{b\alpha}{a}} \left(\int |f(x)|^c dx \right)^{\frac{b\beta}{c}}.
\end{aligned}$$

Take the b^{th} root of both sides to get

$$||f||_b \leq ||f||_a^{\alpha} ||f||_b^{\beta} \implies C = 1$$

which proves the statement. $\square$